

THE ROLE OF CONCEPTUAL SUBITISING IN THE DEVELOPMENT OF FOUNDATIONAL NUMBER SENSE

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Evidence indicates that children with a well-developed number sense are more likely to experience long-term mathematical success than children without. However, number sense has remained an elusive concept. In this paper we summarise the development of an eight dimensional framework categorising what we have come to call foundational number sense, or those non-innate number-related competences typically taught during the first years of schooling. We also show, drawing on grade one lessons from Hungary and Sweden, how teaching focused on conceptual subitising, the teaching of children to identify and use easily recognisable groups of objects to structure children's understanding of number, facilitates students' acquisition of a range of foundational number sense-related competences.

INTRODUCTION

Over the last fifteen years since the publication of Clements' (1999) well-known paper, various scholars, particularly in the United States, have been encouraging teachers to attend to the development of young learners' conceptual subitising (See, for example, Clements & Sarama, 2009; Conderman et al., 2014); where conceptual subitising is the ability to recognise quickly and without counting relatively large numerosities by partitioning these large groups into smaller groups that can be individually subitised (Clements & Sarama, 2007; Geary, 2011). Various claims, which we discuss below, have been made with respect to the efficacy of conceptual subitising-focused instruction. In a related vein, our own recent work has focused on a conceptualisation of foundational number sense (FoNS), which we describe as those number-related competences expected of a typical first grade student that require instruction (Back et al., 2014; Andrews & Sayers, 2014a). FoNS is characterised by eight components, which we describe below. The purpose of this paper, drawing on excerpts from grade one lessons taught by a case study teacher in each of Hungary and Sweden, is to examine the extent to which conceptual subitising-focused activities have the propensity to facilitate students' acquisition of the various FoNS components and, in so doing, examine the warrant for their claimed efficacy.

WHAT IS SUBITISING?

Subitising refers to being instantly and automatically able to recognise small numerosities without having to count (Clements, 1999; Jung et al., 2013; Moeller et al., 2009; Clements & Sarama, 2009). Children as young as three are typically able to subitise numerosities up to three (Fuson, 1988, Moeller et al., 2009), while most

adults are able instantly to recognise without counting the numerosity represented by the dots on the face of a die (Jung et al. 2013). This process, innate to all humans, is typically known as perceptual subitising (Gelman & Tucker, 1975) and forms an element of the preverbal number sense we describe below. In short, perceptual subitising is recognizing a numerosity without using other mathematical processes (Clement, 1999).

Conceptual subitising

However, a second form of subitising, conceptual subitising (Clements, 1999), which is not unrelated to FoNS, has been shown to have considerable implications for teaching and learning. Conceptual subitising relates to how an individual identifies “a whole quantity as the result of recognizing smaller quantities... that make up the whole” (Conderman et al., 2014, p.29). More generally, it can be summarised as the systematic management of perceptually subitised numerosities to facilitate the management of larger numerosities (Obersteiner et al., 2013). For example, when a child is confronted by two dice, one showing three and another showing four, each is perceptually subitised before any sense of seven can emerge.

Subitising can be construed as having a synonymy with the spatial structuring of numbers (Battista et al., 1998). In this case, the ability to recognise and manipulate numbers spatially, through the use of, for example, dice, dominoes and ten-frames, plays a significant role in the development of children’s understanding of both number and arithmetic (Hunting, 2003; Mulligan & Mitchelmore, 2009; Van Nes & De Lange, 2007; Van Nes & Van Eerde, 2010). Indeed, research has shown that conceptual subitising can be taught through mathematical tasks that provide structured images of numbers (Clements, 2007; Mulligan et al., 2006), including fingers to represent small numbers (Penner-Wilger et al., 2007).

In addition to being a powerful tool in the development of children’s general understanding of numbers (Jung, 2011; Penner-Wilger et al., 2007) conceptual subitising has been linked positively to a variety of particular learning outcomes such as counting and counting speed (Benoit et al., 2004) and an understanding of cardinality (Baroody, 2004; Butterworth, 2005; Jung, 2011). Conceptual subitising underpins children's understanding of the equivalence of different decompositions or partitions of numbers (Hunting, 2003; Van Nes & De Lange, 2007), commutativity of addition (Van Eerde, 1996) and the part-whole knowledge (Jung, 2013; Young-Loveridge, 2002) necessary for understanding that $8+6=14$ because $5+5=10$, $3+1=4$ and $10+4=14$ (Van Nes & Doorman, 2011).

Importantly, poor performance on both perceptual subitising (Landerl et al., 2004) and conceptual subitising (Mulligan et al., 2006) may be linked to later mathematical difficulties. In particular, they will be handicapped in their learning of arithmetic (Clements, 1999).

WHAT IS NUMBER SENSE?

Number sense represents a “traditional emphasis in early childhood classrooms” (Casey et al., 2004: 169) and is a key component of many early years’ mathematics curricula (Howell & Kemp 2005; Yang & Li, 2008). However, it has, for many years, remained definitionally elusive (Gersten et al., 2005). As Griffin (2004) noted;

“What is number sense? We all know number sense when we see it but, if asked to define what it is and what it consists of, most of us, including the teachers among us, would have a much more difficult time. Yet this is precisely what we need to know to teach number sense effectively. (p. 173).

Three conceptions of number sense

Our constant comparison analysis (Strauss & Corbin, 1998) of the literature, a process which has been described extensively in Andrews & Sayers (2014b), has identified three distinct conceptions. The first, an innate or preverbal number sense (Butterworth, 2005; Ivrendi, 2011; Lipton & Spelke, 2005), comprises an understanding of small quantities that allows for comparison. For example, children at six months can discriminate numerosities with a 1:2 ratio (Feigenson et al., 2004), while children at 4 can subitise the numerosity of sets containing up to five items (Gelman & Tucker, 1975). These numerical discriminations, are thought to underpin the acquisition of verbal counting skills (Gallistel & Gelman, 2000) and arithmetic (Zur & Gelman, 2004). This preverbal number sense develops in the early years as an innate consequence of human, and other species’ evolution and, importantly, is independent of formal instruction (Dehaene, 2001; Feigenson et al., 2004).

Our second perspective relates to what we have labelled foundational number sense (FoNS). FoNS comprises those number-related understandings that require instruction and which typically occur during the first years of school (Ivrendi, 2011; Jordan & Levine, 2009). It is something “that children acquire or attain, rather than simply possess” (Robinson et al., 2002, p. 85) and reflects, inter alia, elementary conceptions of number as a representation of quantity or a fixed point in the counting sequence (Griffin, 2004). We return to FoNS shortly, but first we summarise our third perspective, which have labelled applied number sense.

Put simply, applied number sense refers to those core number-related understandings that permeate all mathematical learning (Faulkner, 2009, Faulkner & Cain, 2013; National Council of Teachers of Mathematics, 1989). Applied number sense refers to the “basic number sense which is required by all adults regardless of their occupation and whose acquisition by all students should be a major goal of compulsory education” (McIntosh et al., 1992, p3).

DEFINING FOUNDATIONAL NUMBER SENSE

Over the last two years, we have been developing a simple-to-operationalise framework for analysing the FoNS-related opportunities teachers provide for their students. In some respects this remains a work in progress. For example, in its first manifestation (Back et al., 2014), seven components were identified and evaluated against case study teaching in England and Hungary. Two teachers' lessons, focused explicitly on number sequence-related learning, showed that the framework operationalised six of the seven categories – only estimation was missing – and indicated, also, differences in the ways in which the various components interacted in different excerpts analysed. In a second manifestation (Andrews & Sayers, 2014a), seven categories were again discussed, of which six were common to both studies. This second, literature-based, paper offered a detailed account of the development of the seven components. More recently (Andrews & Sayers, 2014b), we have presented a stronger explanatory narrative for the eight component FoNS framework and provide, through the examination of excerpts drawn from the teaching of exemplary teachers in England, Hungary and Sweden, further evidence of the analytical power of the FoNS framework. These eight components are summarised thus:

Number recognition

FoNS aware children are able to recognise number symbols and know their associated vocabulary and meaning (Malofeeva et al., 2004). They can both identify a particular number symbol from a collection of number symbols and name a number when shown that symbol (Clarke and Shinn, 2004; Gersten et al., 2005; Van de Rijt et al., 1999; Yang and Li, 2008). Children who experience difficulty with number recognition experience later mathematical problems generally (Lembke and Foegen, 2009) and particularly with subitising (Koontz and Berch, 1996; Stock et al. 2010). Alternatively, children who recognise numbers are more able to manage multi-digit arithmetic than those who cannot (Desoete et al., 2012; Krajewski & Schneider, 2009). Such skills are better predictors of later mathematics achievement than either general measures of intelligence or earlier achievement scores (Geary et al., 2009), effects lasting as late as adolescence (Geary, 2013).

Systematic counting

FoNS aware children can count systematically (Berch, 2005; Clarke and Shinn, 2004; Gersten et al., 2005; Griffin, 2004; Van de Rijt et al., 1999) and understand ordinality (Ivrendi, 2011; Jordan et al., 2006; LeFevre et al., 2006; Malofeeva et al. 2004; Van Luit and Schopman, 2000). FoNS-aware children count to twenty and back or count upwards and backwards from an arbitrary starting point (Jordan and Levine, 2009; Lipton and Spelke, 2005), knowing that each number occupies a fixed position in the sequence of all numbers (Griffin et al., 1994). Indeed, the skills of symbolic number ordering underpin later arithmetical competence in general

(Gersten et al. 2005; Passolunghi et al., 2007; Stock et al., 2010) and mental arithmetical competence in particular (Lyons and Beilock, 2011).

Awareness of the relationship between number and quantity

FoNS aware children understand the relationship between number and quantity. In particular, they understand not only the one-to-one correspondence between a number's name and the quantity it represents but also that the last number in a count represents the total number of objects, ordinality (Jordan and Levine, 2009; Malofeeva et al, 2004; Van Luit and Schopman, 2000). The correspondence between a number's name or symbol and the quantity represented is, essentially, a human invention requiring instruction (Geary, 2013). Children who have difficulty with this mapping process tend to experience later mathematical difficulties (Kroesbergen et al., 2009; Mazzocco et al., 2011).

Quantity discrimination

FoNS aware children understand magnitude and can compare between different magnitudes (Clarke and Shinn, 2004; Griffin, 2004; Ivrendi, 2011; Jordan et al., 2006; Jordan and Levine 2009; Yang and Li, 2008). They deploy language like 'bigger than' or smaller than' (Gersten et al., 2005), understanding that eight represents a quantity that is bigger than six but smaller than ten (Baroody & Wilkins, 1999; Lembke and Foegen, 2009). Magnitude-aware children have moved beyond counting as "a memorized list and a mechanical routine, without attaching any sense of numerical magnitudes to the words" (Lipton and Spelke, 2005, p. 979). Moreover, magnitude awareness has been shown to be a predictor, independently of ability or age, of more general mathematical achievement (Aunio and Niemivirta, 2010; De Smedt et al., 2009, 2013; Desoete et al., 2012; Holloway and Ansari, 2009; Nan et al 2006; Stock et al. 2010;).

An understanding of different representations of number

FoNS aware children understand that numbers can be represented differently (Ivrendi, 2011; Jordan et al., 2007; Yang and Li, 2008) and that these "act as different points of reference" (Van Nes and Van Eerde, 2010, p. 146). The better children understand a number line, for example, the higher their later arithmetical achievement (Siegler and Booth 2004; Booth and Siegler, 2006, 2008). The better a child understands a partition as a representation of a number, the better developed is that child's later understanding of numerical structures (Thomas et al., 2002) and arithmetical skills (Hunting, 2003). The more competent a child is with regard to the use of fingers in both counting and early arithmetic, skills that can be taught effectively (Gracia-Bafalluy and Noël, 2008), the more competent that child is in later years (Fayol et al., 1998; Jordan et al., 1992; Noël, 2005). Significantly, the use of finger strategies increases as socio-economic status increases, justifying targeted interventions (Jordan et al., 1992; Levine et al., 1992) The use of manipulatives, particularly linking blocks, facilitates counting and the identification of errors (Van

Nes and Van Eerde, 2010). Thus, the better the connections between different representations the more likely a child is to become arithmetically competent (Mundy and Gilmore, 2009; Richardson, 2004; Van Nes and De Lange, 2007, Van Nes and Van Eerde, 2010).

Estimation

FoNS aware children are able to estimate, whether it be the size of a set (Berch, 2005; Jordan et al., 2006, 2007; Kalchman et al., 2001; Malofeeva et al 2004; Van de Rijt et al., 1999) or an object (Ivrendi, 2011). Estimation involves moving between representations - sometimes the same, sometimes different - of number, for example, placing a number on an empty number line (Booth and Siegler, 2006). However, the skills of estimation are dependent on the skills of a child to count (Lipton and Spelke, 2005). Estimation is thought to be a key determinant of later arithmetical competence, particularly in respect of novel situations (Booth and Siegler, 2008; Gersten et al., 2005; Holloway and Ansari, 2009; Libertus et al., 2011; Siegler and Booth, 2004).

Simple arithmetic competence

FoNS aware children can perform simple arithmetical operations (Ivrendi, 2011; Jordan and Levine 2009; Malofeeva et al., 2004; Yang and Li, 2008); skills which underpin later arithmetical and mathematical fluency (Berch, 2005; Dehaene, 2001; Jordan et al., 2007). Indeed, simple arithmetical competence, which Jordan and Levine (2009) describe as the transformation of small sets through addition and subtraction, has been found to be, at grade one, a stronger predictor of later mathematical success than measures of general intelligence (Geary et al., 2009; Krajewski & Schneider, 2009). However, drawing on their experiences of combining physical objects, children's ability to solve nonverbal problems develops before the ability to solve comparable word problems (Levine et al., 1992).

Awareness of number patterns

FoNS aware children understand and recognise number patterns and, in particular, can identify a missing number (Berch, 2005; Clarke and Shinn, 2004; Gersten et al., 2005; Jordan et al., 2006, 2007). Such skills reinforce the skills of counting and facilitate later arithmetical operations (Van Luit and Schopman 2000). Importantly, failure to identify a missing number in a sequence is one of the strongest indicators of later mathematical difficulties (Chard et al., 2005; Clarke and Shinn, 2004; Gersten et al., 2005; Lembke and Foegen, 2009).

In sum, our systematic analysis of the literature identified eight distinct but not unrelated characteristics of FoNS. The fact that they are not unrelated is important as number sense

“relies on many links among mathematical relationships, mathematical principles..., and mathematical procedures. The linkages serve as essential

tools for helping students to think about mathematical problems and to develop higher order insights when working on mathematical problems” (Gersten et al., 2005, p. 297).

In other words, without the encouragement of such links there is always the risk that children may be able to count competently but not know, for example, that four is bigger than two (Okamoto & Case, 1996).

Implications of FoNS-related learning

The quality of a child’s FoNS has substantial implications. On the one hand, a poorly developed number sense has been implicated in later mathematical failures (Jordan et al., 2009; Gersten et al., 2005), while, on the other, research has shown that the better a child’s number sense the higher his or her later mathematical achievements, both in the short (Aubrey & Godfrey, 2003; Aunio & Niemivirta, 2010) and the longer term (Aubrey et al., 2006; Aunola et al., 2004). Moreover, without appropriate intervention, which research shows can be effective (Van Luit & Schopman, 2000), children who start school with limited number sense are likely to remain low achievers throughout their schooling (Aubrey et al., 2006). Basic counting and enumerations skills are predictive of later arithmetical competence in England, Finland, Flanders, USA, Canada and Taiwan respectively (Aubrey & Godfrey, 2003; Aunola et al., 2004; Desoete et al., 2009; Jordan et al., 2007; LeFevre et al., 2006; Yang & Li, 2008), indicating a cross culturally common phenomenon. In similar vein, the ability to identify missing numbers and discriminate between quantities are also predictors of later success (Chard et al., 2005; Clarke & Shinn, 2004; Jordan et al., 2009), as is competence with number combinations (Geary, et al., 2000; 2009; Locuniak & Jordan, 2008). In short, there is evidence highlighting the significance of the different FoNS components in children’s learning of mathematics.

THIS STUDY

In the following, we examine how two teachers, Klara from Hungary and Kerstin from Sweden, both pseudonyms, teach mathematics to their grade one students. We have chosen to scrutinise teaching activities that we construe as likely to facilitate children’s ability to subitise conceptually, although in neither case was conceptual subitising their explicit intention. The focus of our scrutiny has been the extent to which such activities promote FoNS-related opportunities. But first, we describe briefly data capture and other aspects of our methods.

The two sets of lessons from which we draw our excerpts derive from independently conducted studies. Both were captured as part of projects focused on identifying exemplary practice for the purpose of teacher professional development. With respect to data capture, serendipitously similar approaches had been adopted. Teachers were video-recorded in ways that would capture their actions and utterances and both had been filmed over several lessons to minimise the likelihood

of show-piece lessons. Each lesson was viewed repeatedly by at least two authors, allowing us to determine which FoNS components were addressed. In the following we present the results of our application of the FoNS framework to three excerpts, each focused implicitly on the development of conceptual subitising, from each teacher's sequence of lessons. We do not offer any evaluative commentary as our intention was solely to examine how activities focused on conceptual subitising yielded FoNS-related learning opportunities.

THE HUNGARIAN EXCERPTS

Klara's first excerpt

In an early lesson Klara was observed to use domino templates and counters. Working individually, children were asked to use the domino template to represent two numbers that sum to six. She then collated responses and, with each suggestion, placed a prepared domino on the board. As she worked, she encouraged children to be clear in their descriptions by insisting on their using the terms left and right in relation to the domino they described.

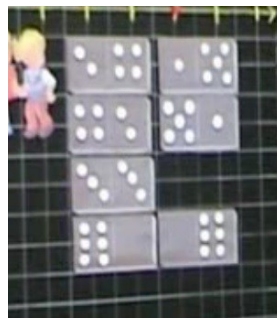


Figure 1: Seven dominoes showing representations of six

The final arrangement can be seen in figure 1. Each domino in each equivalent pair was placed adjacent to the other, although this was not consistently managed, with Klara asking why the double three was lonely. This elicited the response that three on the left and three on the right is the same as three on the right and three on the left.

Commentary on Klara's first excerpt

In this task the dice provided children with familiar arrays to support their subitising of integers up to six. Of course, Klara did not explicitly mention subitising and the more obvious outcome was the use of such arrays, representing familiar subitised numerosities, to allow for significant patterns in numbers to emerge. Interestingly, in comparison with the similar task undertaken by Kerstin, the failure to list the dominoes in numerical order missed an opportunity for greater pattern work, which seemed a rare oversight on Klara's part. From the FoNS perspective, it could be argued that the activity had a clear focus on number patterns, particularly in the ways they combine to make six. There was evidence of different representations of number and, implicitly, evidence of children being asked to undertake simple addition. The

act of linking numbers to the dots was evidence of Klara relating numbers to quantity.

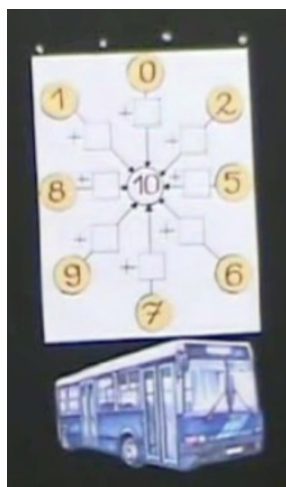


Figure 2: The initial bus trip problems

Klara's second excerpt

The first excerpt began with Klara announcing that her class was to work on a task involving a bus trip and the possibility of ten children going. She revealed the picture shown in figure 2 and, pointing to the zero at the top, asked how many children would be able to join the trip if none were already on the bus. She received the answer of ten and placed the number 10, on a blue background, beneath the figure zero. After a few minutes the picture was, as shown in figure 3, completed with eight different complements to 10 having been added to the diagram. In so doing she had encouraged her students to see every pair of complements with two pairs, $1 + 9$ and $9 + 1$, and $2 + 8$ and $8 + 2$, being repeated. Throughout the process Klara encouraged her students to use their hands. In particular she discussed how the fingers of two hands can be used to represent ten before demonstrating, with respect to two children already on the bus, that if both hands are held open and two fingers are closed - representing the two children on the bus - the remaining fingers represent the number of additional students allowed to travel. Thus, three fingers on one hand and five on the other, drawing on subitised numerosities, facilitate a structural awareness of eight as the sum of five and three.

Commentary on Klara's second excerpt

In the above, Klara did not make explicit the relationship between finger use and subitising, but we believe it is there. The explicit act of, say, closing two fingers and then observing that three plus five fingers remain, while not disallowing the possibility of counting as a strategy, clearly encourages children to focus on the structural relations and the immediate recognition of subitised numerosities. With regard to FoNS, at least four categories were evident. The picture around which Klara structured the different tasks encouraged number recognition and the relationship of number to quantity. Students' use of fingers was an exploitation of

different representations of number, while simple addition, possibly subtraction, was the explicit focus. There were also opportunities for those children who preferred to do so to exploit systematic counting as an additive strategy.

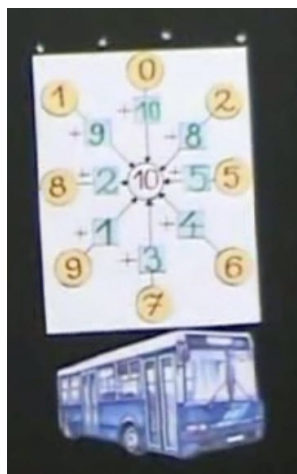


Figure 3: The completed bus trip problems

Klara’s third excerpt

Klara revealed two pictures adjacent to each other, one showing eight girls and the other five boys, before posing the question, how many children would there be in total? Before moving to the solution, Klara, in response to students’ answers to her questions, wrote underneath the pictures

$$8_{gy} + 5_{gy} = ?_{gy}.$$

In this instance, ‘gy’ was an agreed abbreviation of gyerekek (children). After this she turned attention to the eight girls, which she represented by placing eight red counters on a ‘board’, as shown in figure 4. Next, turning to the five boys, she asked, how many would be needed to complete the ‘board’? She received an answer of two, and added them to the board in blue. Next, asking how many boys remain, she was told three and completed the picture as in figure 5. Finally, she asked how many children were there in total, and was told thirteen. With further probes she elicited the result that the thirteen was a result of ten plus three.

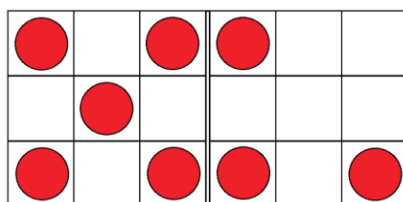


Figure 4: Klara’s placing of eight red counters on a ‘board’

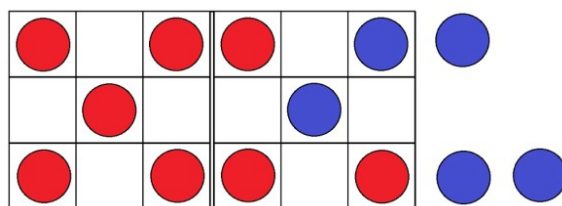


Figure 5: Klara's complete representation of $8 + 5 = 13$

Following this, Klara revealed the next variation of the task, which, as shown in figure 6, showed the 'board' for 8 plus 3. Klara's questions led to her class to agree that the 'board' showed a representation of $8 + 2 + 1$, which was $10 + 1$ or 11.

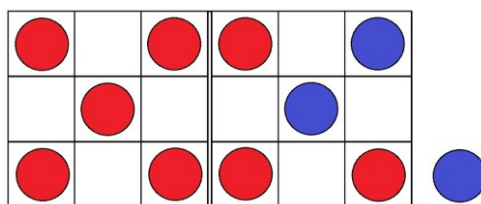


Figure 6: The completed board for $8 + 3 = 11$

This discussion led to the following being written beneath the image

$$8 + 2 + 1 = 11$$

and

$$8 + 3 = 11$$

This process was repeated, exactly as above, for $8+4$ and $8+6$. In all three cases the class chanted the process. For example, 'eight plus two equals ten; ten plus one equals eleven'

Commentary on Klara's third excerpt

This fourth excerpt was more obviously focused on conceptual subitising than the earlier. Klara's focus, in her use of the board, was the representation of ten as two fives. In so doing, she was drawing on familiar subitised numerosities. That is, her students were familiar with and able to recognise the properties of fiveness, and were comfortable with ten as the juxtaposition of two fives. Thus, the addition of five to eight, drew, essentially, on familiar subitised numerosities of five or less. In so doing, she addressed several components of FoNS, not least of which was the explicit focus on simple addition. Her formulation beneath the two pictures encouraged number recognition. The use of the board and its associated counters provided another representation of number and an explicit link between number and quantity. It could also be argued that the use of counters, and the familiar representation of five, was an encouragement for students to explore arithmetical procedures as structural patterns. Interestingly, and unrelated to FoNS, there was even an early introduction to algebraic symbolism and subscripts in her use of *gy* as an abbreviation for the word children.

THE SWEDISH EXCERPTS

Kerstin's first excerpt

Each child in the class was given a small bowl containing six small pebbles and a sheet of paper, laid landscape on the desktop. The paper was halved by means of a pen or pencil laid vertically down the centre of the sheet. Kerstin asked her students to take their six pebbles and, in a way of their choosing, place some on one side of the divide and the others on the other. The only rule is that all six must be used.

While they are doing this, Kerstin attached a metre rule to her whiteboard to create two distinct halves in the same way as her students. She wrote six at the top before inviting her students for, in essence, different partitions of six. A child volunteered three and three. Kerstin placed, towards the vertical middle of her board, three disks to one side of her line and three to the other.

The process continued, a second child suggested two and four. Kerstin placed these above the previous. A third child suggested five and one, and it now becomes clear that Kerstin was placing her counters in such a way that each ordered pair had a well-defined place on her board, with left 0, right 6 at the top coming down to left six, right zero at the bottom. The fourth child suggested six and zero, the next four and two, the next zero, six, before the final child offered one and five. Thus, seven sets of counters had been placed systematically on the board, with each pair, representing a partition of six, in a well-defined position. At this stage Kerstin drew a horizontal line across the board to separate each pair to create the effect shown in figure 7.

Kerstin asked her class how many ways they could make six in this way and received the answer, sju, seven.

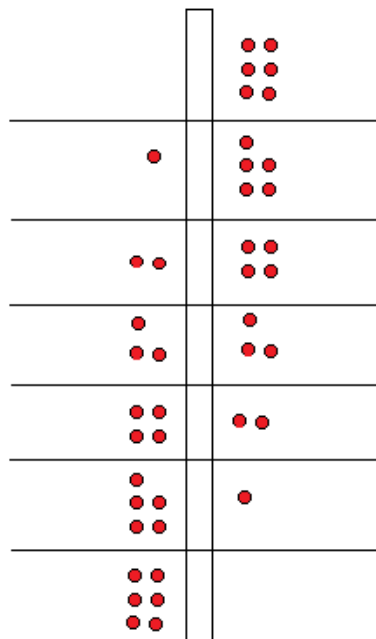


Figure 7: Kerstin's completed diagram

Commentary on Kerstin's first excerpt

We would argue that this activity clearly encouraged students to engage in conceptual subitising. The breaking down of six into different additive pairs, most of which were amenable to perceptual subitising, allowed students to instantly see six. From the perspective of FONS, several categories were evident. Firstly, the use of pebbles served to remind children that numbers represent quantities. Secondly, the various partitions of six allowed children to see different representations of the same number. Thirdly, if only implicitly, the same act of partitioning encouraged children to engage with simple addition. Fourthly, the manner in which Kerstin arranged the solutions on the board highlighted two forms of number pattern. On the one hand there was the clear distinction between the patterns formed by even and odd integers, although, of course, this could also be construed as another perspective on the representation of numbers. On the other hand, the sequencing of the solutions highlighted the fact that as one set of numbers decreases, the other increases. Fifthly, the arrangement of the partitions on the board could be construed as an encouragement for children to see numbers as having well ordered places in the sequence of all numbers as part of a drive to facilitate their counting competence.

Kerstin's second excerpt

Kerstin invited the class to play a game in pairs. One child would take the six pebbles and, behind his or her back, distribute them between his or her two hands. He or she would then reveal one hand's contents and the other child had to say what was in the closed hand. Then the pair would swap roles and repeat the process. Thus, many opportunities were given for children to rehearse the partitions of and complements to six. During this time, Kerstin circulated the room, asking student pairs questions like, if I have two in one hand, how many do I have in the other? At the end of this episode, Kerstin alerted her students to the symmetry of the arrangements on the board by pointing out the connections between 4 plus 2 as well as the 2 plus 4, and the same for 1 and 5 and 5 and 1. She also reminded her students, by moving counters from right to left, that each row of her table summed to six.

Commentary on Kerstin's second excerpt

As with her first activity, it seems to us that this task was focused on the development of conceptual subitising. When circulating the room, Kerstin's questions, focused on the complements to six, drew on children's mental representations of perceptually subitised numerosities like two and four. When summarising the task and its relationship to what was on the board, her explicit matching of, say, four with two and two with four further supported six as a conceptually subitised construct. With respect to FONS-related opportunities, the game allowed children to consolidate the connection between number and quantity and, of course, cardinality as a component of systematic counting. It presented different representations of number and, implicitly, exploited simple arithmetic or

counting to locate missing numbers. Also, in alerting her children to the symmetry of the relationship, she was encouraging an awareness of pattern, even if she was not exploring missing values.

Kerstin's third excerpt

At the start of this third excerpt Kerstin distributed a worksheet (figure 8) to each child. On the worksheet were eight drawings of pairs of hands. One hand was open and showed some pebbles, the other hand was closed to represent the hidden pebbles. As can be seen, the pictures alternated with respect to which hands were open and which closed. Students were invited to find the missing number of pebbles and write the answer in beneath the relevant hand. There were four pairs for six and four for seven.

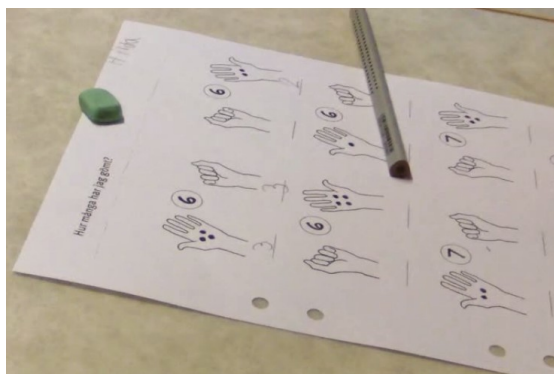


Figure 8: Kerstin's worksheet

While children worked Kerstin circulated and helped those in need. This typically involved her modelling a situation with pebbles in her own hands. On later occasions, particularly when moving working on tasks involving seven, students were encouraged to model the situation as in figure 9.



Figure 9: The subitising model encouraged by Kerstin

Commentary on Kerstin's third excerpt

While it is conceivable that some students may have employed a counting-on strategy, this final activity offered an opportunity for students to consolidate their conceptual subitising of six. This could have been achieved either by means of a mental representation of perceptually subitised numerosities or the use of pebbles as in figure 10. With respect to seven, a number less amenable to perceptual subitising, Kerstin's encouragement to use the pebbles was an explicit encouragement of a

conceptual subitising strategy. From the perspective of FoNS, the exercise consolidated the connection between number and quantity and cardinality as a component of systematic counting. It also presented different representations of number and, implicitly, exploited simple arithmetic or counting to locate missing numbers. Also, students were encouraged to engage in number recognition.

DISCUSSION

In this paper we set out to examine how activities focused on conceptual subitising have the potential to facilitate children's acquisition of the various components of foundational number sense (FoNS). Interestingly, *fons* is the Latin word for fount or spring, which seems apposite for such an important underpinning mathematical understanding.

Our analyses, summarised in table 1, indicate that in both cases, Klara from Hungary and Kerstin from Sweden, a significant proportion of the eight FoNS components were identified in each examined excerpt. In neither set of excerpts was there evidence of quantity discrimination or estimation, although it is probably unrealistic, or even unreasonable, to assume all FoNS components to be addressed in every sequence of activities as some are more likely to lend themselves to particular components than others. However, the number of FoNS components identified in each excerpt - consistently between four and five - indicates that the claims made for the efficacy of teaching focused on conceptual subitising (Clements, 1999; Conderman et al., 2014; Sadler, 2009), are not without warrant.

		Excerpts					
		H1	H2	H3	S1	S2	S3
FoNS component	Number recognition		X	X			X
	Systematic counting		X		X	X	X
	Relating number to quantity	X	X	X	X	X	X
	Quantity discrimination						
	Different representations	X	X	X	X	X	X
	Estimation						
	Simple arithmetic	X	X	X	X	X	X
	Number patterns	X		X	X	X	

Table 1: FoNS-related summary of the various excerpts

It is also interesting to note that in neither case was conceptual subitising an explicit intention - neither teacher was aware of the term - nor were teachers expecting to address FoNS categories of learning. It is also interesting to note that despite

substantial differences in the management of their lessons - Klara spent all her lesson orchestrating whole class activity with only occasional expectations of students working individually, while Kerstin spent the great majority of her time managing and supporting students working individually - the FoNS components addressed in their respective excerpts were remarkably similar.

Finally, in an earlier paper, in which FoNS categories were applied to tasks focused on mathematical sequences, similar results were obtained for Klara, in Hungary and Sarah, in England (Back et al., 2014). That is, the evidence of the two analyses suggests that tasks focused on sequences and tasks focused on conceptual subitising appear rich in their potential for realising a range of FoNS components. Without wishing to overstate the significance of these results, results yielded by small case studies, it is worth suggesting that a potentially exciting line of future enquiry would be to identify other topics and ways of teaching them with similar potential for FoNS developments.

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